

Modern Portfolio Theory

Diversified Portfolio

Portfolio risk comes from two places, systematic risk and idiosyncratic risk. Systematic risk is what costs you money when the market suffers a broad based decline. There's not much you can do to hedge against broad based market moves, but you can plan your portfolio to lay off some of the issue specific risk if you know something about how the issues play against each other. You often hear that diversified portfolios are less susceptible to losses. Sector funds are the antithesis of diversification because they place all your bets on the same breed of dog. A sector fund comprised of forty stocks is more insulated from risk than a portfolio of one stock, but it isn't a particularly diversified portfolio. Gold stocks typically all tank and rally together. Oil stocks track with global oil prices. Durable goods vendors feel the pinch of high interest rates as a group, and banks all rally when there's more room between saving and lending rates. The trick is to have some Christmas and some Fourth of July stocks. Similarly, it's wise to have some stocks that aren't exposed to blizzards and some that aren't exposed to hurricanes; some that rally when capital is cheap (like durable goods manufacturers) and some that rally when capital is dear (like banks). That's common sense in action. It is the layman's approximation to a more analytical approach to diversification.

Covariance and Correlation

We're going to need the concept of correlation shortly. I like to break it down, so it doesn't appear as daunting. First, we need the covariance, defined

$$Cov(r_A, r_B) = \frac{1}{T-1} \sum_{t=1}^T (r_{A,t} - \bar{r}_A)(r_{B,t} - \bar{r}_B)$$

at right, where the r_t are the A and B stocks'

returns through the time period and the \bar{r} represent their respective means.

$$\sigma = \sqrt{\frac{1}{T-1} \sum_{t=1}^T (r_t - \bar{r})^2}$$

The problem with covariance is that it's a number that isn't easily compared with other covariances, so it's convenient to scale it by the standard deviation of the respective returns. The standard deviation (often used as the volatility) is given by the expression at right, using

$$Corr(r_A, r_B) = \frac{Cov(r_A, r_B)}{\sigma_A \sigma_B}$$

the same definitions for r_t and \bar{r} . The correlation of two stocks' returns can therefore be defined as:

Uncorrelated and Anti-correlated Portfolios

If you have access to historical prices, e.g. from [Google Finance](http://finance.google.com/),¹ [Yahoo! Finance](http://finance.yahoo.com/),² or [MSN's MoneyCentral](http://moneycentral.msn.com/)³ you can pursue strategies that optimize returns at minimum risk, without delving as far into the business' susceptibility to hurricanes or interest rates. You need to know less about those risks if you can find complementary behavior in their historical market prices. Correlation measures the degree of similarity between data sets. Correlation is defined by the formula above with the resulting r always landing between -1 and +1. The closer to +1, the more the sequences look alike. The closer to 0, the less they have in common. The nearer to -1 the more they complement each other. A portfolio with all 0 inter-issue correlations means that the issues' market returns have absolutely no mathematically discernable relationship - they are perfectly uncorrelated.

The best possible risk management for a two stock portfolio (with an equal cost basis in each issue) is achieved with a correlation of -1, where the issues are perfectly anti-correlated. That's means that market value lost by one of the issues occurs exactly in lockstep with an increase in the market value of the other. The parenthetical qualifier is a hint that the cost-basis is a factor. What you really want

¹ <http://finance.google.com/>

² <http://finance.yahoo.com/>

³ <http://moneycentral.msn.com/>

satisfies: $Corr(r_A, r_B)C_1 + (1 - Corr(r_A, r_B))C_2 = 0$ where the C_i represent the cost basis of each issue. This ideal can be applied to pairs trading, but becomes unwieldy as more issues are included in the portfolio. Fortunately, there are theories to address the general case.

The Capital Asset Pricing Model

The first step was probably Markowitz' Efficient Portfolio, which addressed diversification, risk, and return. It laid the groundwork for the Capital Asset Pricing Model (CAPM) which separated systematic and idiosyncratic risk factors, and identified the efficient frontier. Systematic risk derives from broad market moves and cannot be diversified away.

$$P = \begin{bmatrix} Corr(r_A, r_A) & Corr(r_B, r_A) & Corr(r_C, r_A) \\ Corr(r_A, r_B) & Corr(r_B, r_B) & Corr(r_C, r_B) \\ Corr(r_A, r_C) & Corr(r_B, r_C) & Corr(r_C, r_C) \end{bmatrix}$$

Idiosyncratic risk is attributable to market price correlations - it can be diversified away by selecting uncorrelated or anti-correlated issues. The efficient frontier is a curve that identifies the optimal return at a given risk tolerance. The Wikipedia has an excellent article on the [CAPM](#).⁴ Of particular note are the assumptions and consequent shortcomings.

$$P = \begin{bmatrix} 1 & Corr(r_B, r_A) & Corr(r_C, r_A) \\ Corr(r_A, r_B) & 1 & Corr(r_C, r_B) \\ Corr(r_A, r_C) & Corr(r_B, r_C) & 1 \end{bmatrix}$$

The Wikipedia page uses algebraic notation to discuss the theory, but I think a vector representation presents the general case better. First, we need the portfolio's correlation matrix P . The correlation of any signal with itself is 1, hence the second representation of P above. Using P , we construct the portfolio variance-covariance matrix Σ , as follows:

$$\Sigma = \begin{bmatrix} \sigma_A & 0 & 0 \\ 0 & \sigma_B & 0 \\ 0 & 0 & \sigma_C \end{bmatrix}^T \begin{bmatrix} 1 & Corr(r_B, r_A) & Corr(r_C, r_A) \\ Corr(r_A, r_B) & 1 & Corr(r_C, r_B) \\ Corr(r_A, r_C) & Corr(r_B, r_C) & 1 \end{bmatrix} \begin{bmatrix} \sigma_A & 0 & 0 \\ 0 & \sigma_B & 0 \\ 0 & 0 & \sigma_C \end{bmatrix}$$

... which becomes:
$$\Sigma = \begin{bmatrix} \sigma_A \sigma_A & \sigma_B Corr(r_B, r_A) \sigma_A & \sigma_C Corr(r_C, r_A) \sigma_A \\ \sigma_A Corr(r_A, r_B) \sigma_B & \sigma_B \sigma_B & \sigma_C Corr(r_C, r_B) \sigma_B \\ \sigma_A Corr(r_A, r_C) \sigma_C & \sigma_B Corr(r_B, r_C) \sigma_C & \sigma_C \sigma_C \end{bmatrix}.$$

Given $Corr(r_A, r_B) = \frac{Cov(r_A, r_B)}{\sigma_A \sigma_B}$ and $Var(r) \equiv \sigma^2$, we can rewrite Σ , as:

$$\Sigma = \begin{bmatrix} Var(r_A) & Cov(r_B, r_A) & Cov(r_C, r_A) \\ Cov(r_A, r_B) & Var(r_B) & Cov(r_C, r_B) \\ Cov(r_A, r_C) & Cov(r_B, r_C) & Var(r_C) \end{bmatrix}$$

The variance-covariance matrix is a factor in the portfolio variance σ_P^2 :

⁴ http://en.wikipedia.org/wiki/Capital_Asset_Pricing_Model

$$\sigma_p^2 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^T \begin{bmatrix} \text{Var}(r_A) & \text{Cov}(r_B, r_A) & \text{Cov}(r_C, r_A) \\ \text{Cov}(r_A, r_B) & \text{Var}(r_B) & \text{Cov}(r_C, r_B) \\ \text{Cov}(r_A, r_C) & \text{Cov}(r_B, r_C) & \text{Var}(r_C) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

... and the standard deviation σ_p of its variations about the mean, in vector form: $\sigma_p = \sqrt{\vec{w}^T \Sigma \vec{w}}$. The standard deviation of stock returns is presumed to be a good model of stock volatility.

Efficient Frontier

The minimum risk portfolio can be found by minimizing σ_p over w , where the sum of the weights must be 1. If also constrained by $\mu_0 \leq \vec{w}^T \vec{\mu}$, where μ_0 is the minimal acceptable return, you can recover the portfolio volatility at μ_0 by computing $\sigma_p = \sqrt{\vec{w}^T \Sigma \vec{w}}$ using the weight vector \vec{w} found during optimization. The [efficient frontier](#)⁵ is a curve defined by a series of such points in volatility-return space.

Sharpe Ratio

Another of Sharpe's contributions to computational finance is the [Sharpe ratio](#),⁶ sometimes called the market price of risk. After a 1994 revision, it looks like the ratio at right, which divides the expected excess return (or premium over the risk free rate) by the standard deviation (square root of the variance) of the excess return. The goal of the CAPM is to find the point on the efficient frontier where the Sharpe ratio is maximized.

$$S_p = \frac{E[R_p - R_f]}{\sqrt{\text{var}[R_p - R_f]}}$$

Tangency Portfolio

As it turns out, the Sharpe ratio is maximized at the point where the efficient frontier is tangent to the [Capital Market Line](#)⁷ which also passes through the risk free rate $(\sigma, \mu) = (0, R_f)$ in volatility-return space. The tangency point represents a portfolio that achieves the ideal balance of risk free assets (e.g. T-Bills, represented by the Capital Market Line) and risky assets (e.g. stocks represented by the efficient frontier). The investment objective of balanced funds is typically to hold a tangency portfolio. Sharpe's quest for an analytical solution to the tangency point ultimately led him to develop Modern Portfolio Theory. Campbell Harvey, at Duke University, uses Lagrange multipliers to find the tangency point and posted this [solution](#)⁸ to the Capital Market Line: $R_p = R_f + \sqrt{\kappa} \sigma$, where

$\kappa = (\vec{\mu} - R_f \vec{1})^T \Sigma^{-1} (\vec{\mu} - R_f \vec{1})$ and $\vec{\mu}$ a vector of average returns for each of the issues in the portfolio.

Beta

⁵ http://en.wikipedia.org/wiki/Image:Markowitz_frontier.jpg

⁶ http://en.wikipedia.org/wiki/Sharpe_ratio

⁷ Also known as the Capital Allocation Line and less frequently as the Capital Asset Line.

⁸ http://www.duke.edu/~charvey/Courses/ba350_1997/diverse/diverse2.htm

The covariance of the market with a portfolio is known as the portfolio's Beta, $\beta_p = \rho_{p,m} \frac{\sigma_p}{\sigma_m} = \frac{\sigma_{p,m}}{\sigma_m^2}$

where $\rho_{p,m}$ is the correlation between the portfolio and the market, σ_p is the portfolio's historical volatility (really standard deviation), and σ_m is the market's historical volatility.

Expected Return

According to the CAPM, your expected return will be $E[R_p] = R_f + \beta_p (E[R_m] - R_f)$ at the tangency point. Note that the portfolio's volatility is not obviously a factor – it's buried in the portfolio beta β_p which scales the portfolio's exposure to market volatility. All expectations are identically distributed, so we can say: $R_p = R_f + \beta_p (R_m - R_f) + e_p$, where e_p is the deviation from the expected value of R_p .

When the CAPM assumptions are taken to their logical conclusion, all investors will hold the identical market portfolio. Of course, this is impossible, due to the difficulty of being so widely diversified, fractional share issues, the need to continuously trade to rebalance the portfolio, etc.

Modern Portfolio Theory

Building on the CAPM, Modern Portfolio Theory incorporates borrowing and short selling, thus the possibility of returns above the efficient frontier of the CAPM. The Wikipedia article on [Modern Portfolio Theory](http://en.wikipedia.org/wiki/Modern_portfolio_theory)⁹ also merits review.

To minimize risk (volatility) at a given level of expected return, the optimization problem is:

$\min \sqrt{\vec{w}^T \Sigma \vec{w}}$ subject to: $\vec{w}^T \vec{r} = \check{r}$ and $\vec{w}^T \vec{1} = 1$, where \check{r} is the desired return. The Lagrangian function looks like: $\vec{w}^* = \lambda_1 \Sigma^{-1} \vec{r} + \lambda_2 \Sigma^{-1} \vec{1}$, where

$$\lambda_1 = \kappa_1^{-1} - \lambda_2 \kappa_2 \kappa_1^{-1},$$

$$\lambda_2 = (\check{r} \kappa_3^{-1} - \kappa_1^{-1}) (\kappa_1 \kappa_3^{-1} - \kappa_2 \kappa_1^{-1})^{-1},$$

$$\kappa_1 = \vec{1}^T \Sigma^{-1} \vec{r},$$

$$\kappa_2 = \vec{1}^T \Sigma^{-1} \vec{1},$$

$$\kappa_3 = \vec{r}^T \Sigma^{-1} \vec{r}, \text{ and}$$

$$\kappa_4 = \vec{r}^T \Sigma^{-1} \vec{1}.$$

Due to symmetry about the diagonal of the variance-covariance matrix, $\kappa_1 = \kappa_4$.

With the CAPM as its foundation, Modern Portfolio Theory suffers from many of the same shortcomings. The Normal distribution, a central assumption of the CAPM, is analytically tractable, but imposes up and down-side symmetry on market behavior. The Sharpe ratio consequently grades portfolios with a strong upside return and a mild downside exactly like its downside converse.

⁹ http://en.wikipedia.org/wiki/Modern_portfolio_theory

Bibliography

Frank Fabozzi, Petter Kolm, Dessislava Pachamanova, Sergio Focardi, Robust Portfolio Optimization and Management, 2007, John Wiley & Sons, Hoboken, NJ.

Mikkel Rasmussen, Quantitative Portfolio Optimisation, Asset Allocation and Risk Management, 2003, Palgrave Macmillan, New York, NY.